CLIFF LONG

Bug Bands and Monkey Saddles

It was almost thirty years ago that I carved "a bug on a band" (fig. 1) patterned after a sculpture seen as a boy at the Museum of Science and Industry in Chicago. It was carved from a solid piece of white pine, to illustrate that a Möbius band is one-sided. It is slotted in the middle to allow movement of the lady bug. Once around a circular path and the bug is upside down; twice around and it is right-side up again. Such a band (or surface) is called *non-orientable*. This band also has only one edge and the band itself is called a *spanning surface* of that edge curve, meaning that the surface is bounded by the curve. The edge in this case can be continuously transformed to the shape of a circle (without cutting or piecing).

Over the years, I've been quite interested in computer graphics and computer aided design, and even though the current 3D graphics packages give fantastic images, I frequently find a solid model to be very helpful in gaining a real feeling (pun intended) for the surface under consideration. Two specific surfaces come to mind—those studied by Art Winfree and David Hoffman which will be mentioned later. But often, I just enjoy the creation of an object whose shape appeals to me for one reason or another—mathematically, artistically or just for fun.

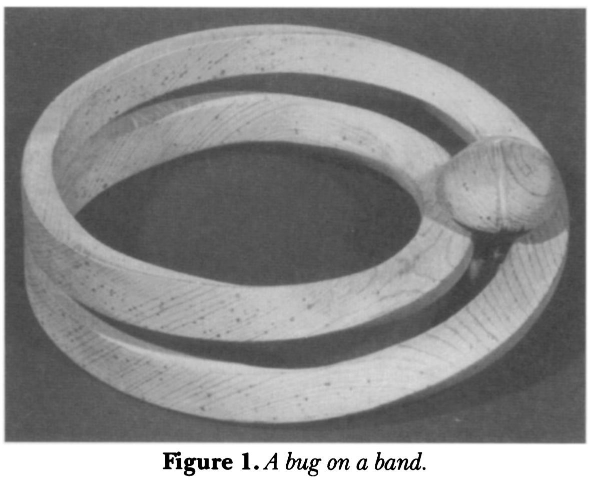

Figure 1. *A bug on a band.*

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Computer Controlled Carving

My interest in computer-aided geometric design began with a consulting project which a colleague, Vic Norton, and I had with Ford Motor Company. This project introduced us to a computer numerically controlled milling machine which is a very convenient tool for simulating functions of two variables in a wooden, wax or aluminum block. I actually have a small milling machine in my office which I occasionally use to create solid models to illustrate mathematical ideas for my students. Three samples are shown in fig 2: a "monkey saddle", a numerical solution of a partial differential equation, and an upside down modulus surface for a cubic polynomial [6]. The milling machine which I have is a mini three axis prototype designed and built by a former student, Bruce Ottens. Many schools now have them available in the technology area, and often are quite willing to share them, even with mathematicians, for interesting projects. This type of carving does, however, require the ability to describe the surfaces mathematically. Often my hand carvings are motivated simply by a picture or verbal description of an object.

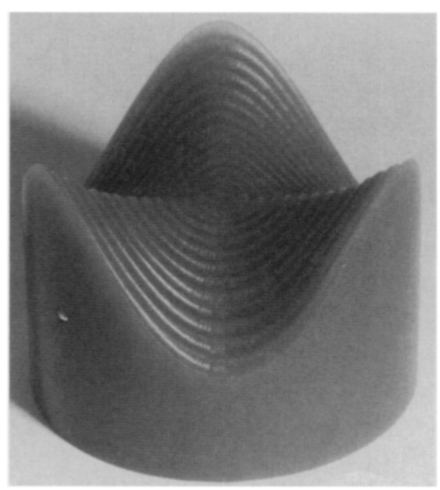

Figure 2. (a) *a "monkey saddle".*

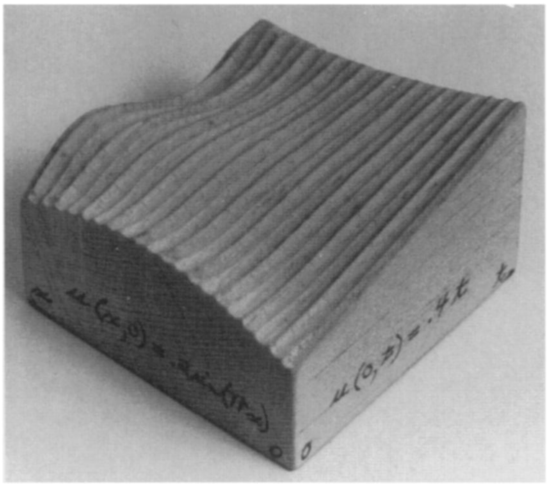

Figure 2. (b) *a numerical solution of a partial differential equation.*

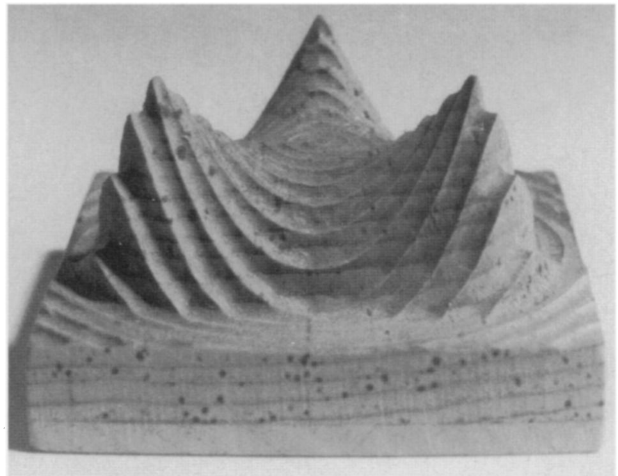

Figure 2. (c) *an upside down modulus surface for a cubic polynomial.*

Surfaces From Knots

A knot is just a curve in three space which is homeomorphic to a circle, i.e., the points on the knot and circle can be placed in one-to-one correspondence with continuity in both directions. The edge of a Möbius band is the simplest example (sometimes called the unknot), whereas, the trefoil knot is the simplest non-trivial knot. More involved knots and their spanning surfaces may frequently be seen in the magnificent art of sculptors Charles O. Perry [12] and David Chamberlain [1]—often seen on display in front of public buildings and as high as ten to thirty feet. While they don't claim to be mathematicians, a mathematical bent is certainly evident in their work.

Stereo views help, but ...

Only recently, upon seeing computer stereo images of a spanning surface for a trefoil knot in an article by A. T. Winfree on excitable media [15, p.15], I again resorted to a wood carving to gain an understanding of the surface. The stereo images are shown in fig. 3. It is quite possible to see depth inherent in these views without the aid of a viewer. (Simply get each eye looking at its own view. A distance of about 15 inches is best and some people find the placement of a 3 × 5 card on edge between the eye views to be helpful. A critical restriction for stereo viewing is to keep your eye level parallel to the picture horizontal. Alternately closing the left and right eyes sometimes helps in homing in on the depth cue.) I thought that I had a good idea of what the surface looked like, but I realized partway through the carved model that I really did not. A flow pattern, which seemed rather evident in the finished carving of fig. 4, was not at all evident in the stereo views, due in large part to the particular parametrization used in the wire frame stereo views. The inherent flow of this carving encouraged me to return to the *Topological Picture Book* by George Francis [5], and later to the book *Knot Theory*, by Charles Livingston [9] for more recent information on knots and spanning surfaces.

In the meantime I was teaching third semester calculus, where we were encountering the Frenet frame (unit tangent, normal and binormal vectors to a curve at a point) determined via the tangent, velocity, and acceleration vectors to a curve in 3-D (such as our trefoil knots). As a result, it seemed natural to twist a band centered on a trefoil knot—just to see how much the Frenet frame twists in space for this knot. The band was formed by rectangles placed perpendicular to the curve with their axes in the directions of the normal and binormal vectors. My colleague, Tom Hern, and I created stereo views of the band (fig. 5) using a computer graphics package called RayShade. The left two views in fig. 5 are used for those viewers who prefer cross eyed viewing and the right two views are for those who prefer straight ahead view-

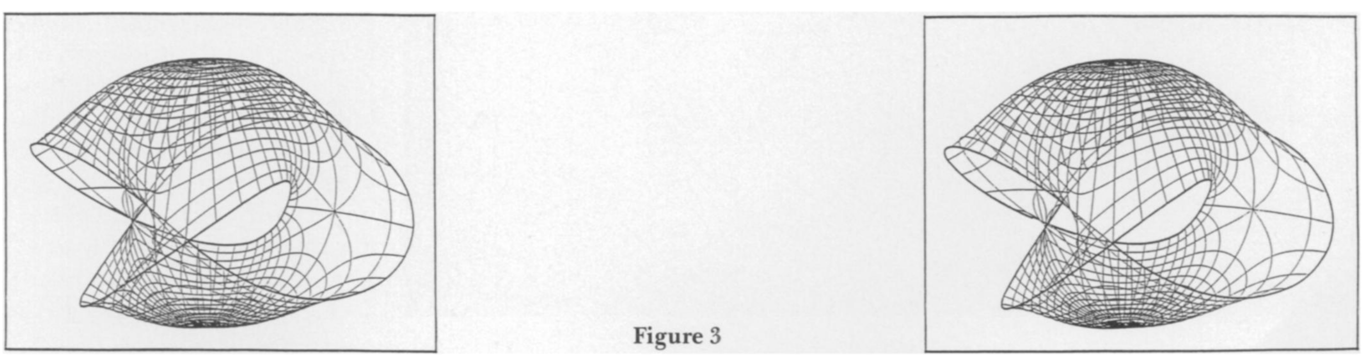

Figure 3

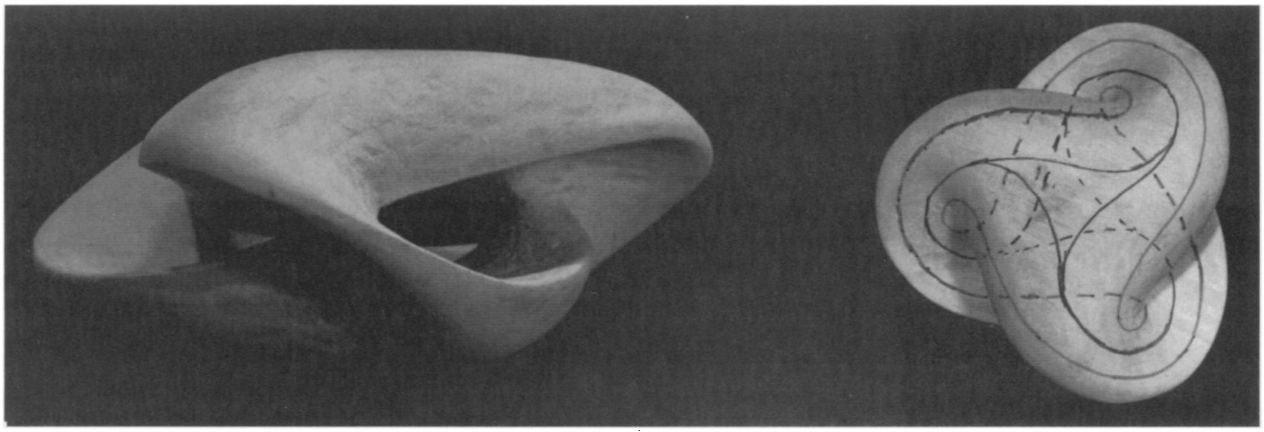

Figure 4

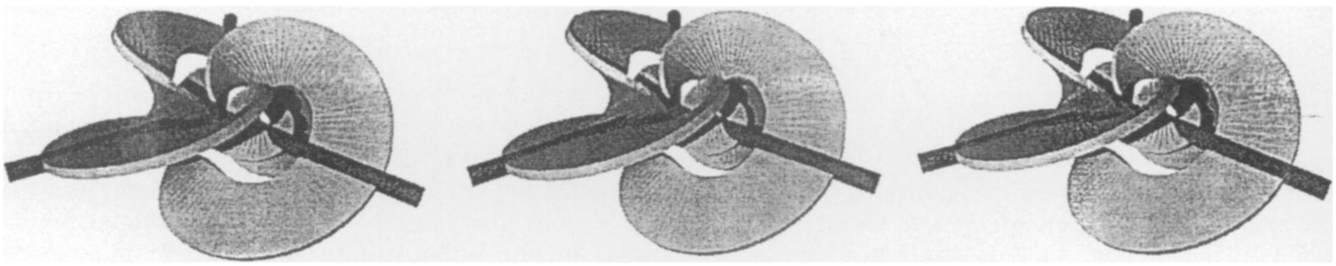

Figure 5. *Left pair for cross-eyed viewing and right pair for wall-eyed ciewing.*

ing. After viewing several widths of such bands, we observed that expanding the width of the surface (shown in fig. 6a) would allow it to self-intersect in a new surface of fig. 6b, which entailed another wood carving to aid in understanding the resulting surface shape. This hand carved model did not require a mathematical formula, however, the parametrized descriptions of its predecessors were necessary for the model of fig. 6a whose creation will now be described.

Stereolithography Models

The twisted trefoil knot surface shown in fig. 6a was created on a 3D Systems stereolithographic machine [16]. It is formed from a tub of light sensitive polymer by focusing a laser beam on a thin layer of the polymer and hardening just that portion that is needed for this level of the resulting model. Then a new layer of polymer is exposed above the first in order to form the next level of the model. The model shown is made up of many levels and required several hours to create—but once described and started, requires no operator intervention in the forming process. For further information and examples of some outstanding surfaces created via stereolithography, I recommend the work of Stewart Dickson described in [13]. Stereolithography is one of the recent techniques used in the rapid prototyping industry (which accounts for a significant shortening of the lag time between computer aided design and a finished product) and is still rather expensive for casual modeling. The following quote is from the publication, the *Edge* [17], put out by 3D Systems [16], and indicates a very important application. "Stereolithography models enable surgeons to rehearse difficult surgeries and create cutting templates and exact bone grafts before the patient ever enters the operating room.... helping surgeons cut the operating time up to thirty percent." CAT scan information is frequently used to provide the information needed for each level of the resulting medical model.

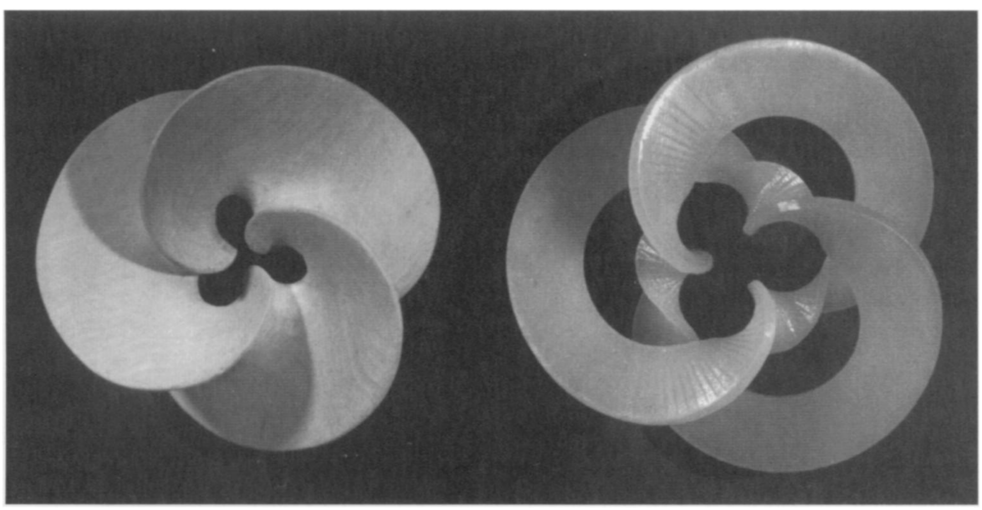

Figure 6

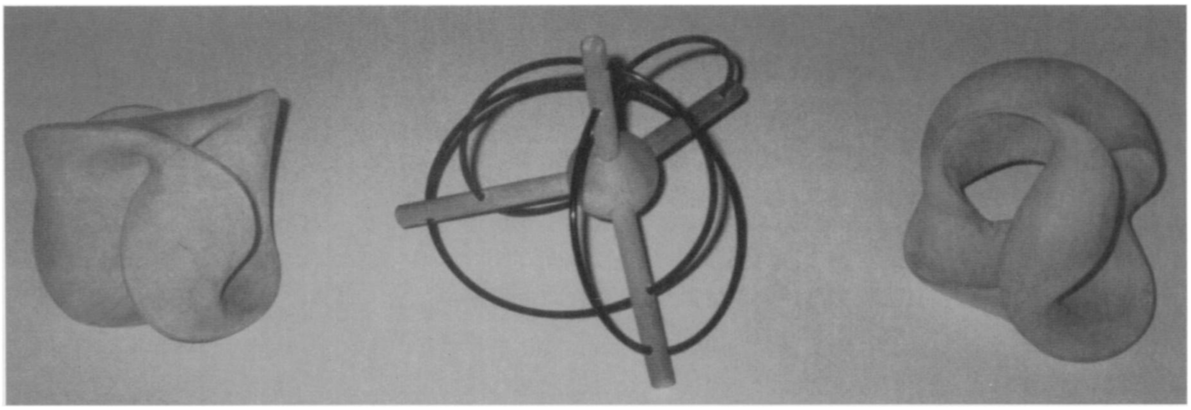

Figure 7

Models From Soap Film

It is known [9, p. 70] that every knot has at least one orientable spanning surface. Many of these are stable enough that they can be generated using soap film and wire frame models of the knots. These minimal surfaces are often quite beautiful, but short-lived. Some tips on creating and prolonging the images can be found in the article by Richard Courant [2]. After many dippings and viewings, a figure-eight knot (four-knot) managed to yield a surface similar to that shown in the wooden sculpture of fig. 7a—an orientable spanning surface of the four knot shaped much like the one in fig. 7b. The terminology "figure eight knot" is the common terminology of sailors and others who use this knot to create a large knob on the end of a rope to keep it from slipping through a pulley. The mathematical terminology "four knot" denotes the smallest number of crossings visible by viewing the knot from any direction. Returning briefly to fig. 4, it is clear that the surface there is not a minimal surface since it bulges out in places, and the question arises as to the shape of a similar type artistic spanning surface for a figure-eight knot. The proud sculptor is shown with this his favorite sculpture in fig. 8. This non-orientable model is also shown in fig. 7c where it can be compared to its defining knot. A movie resulting from storing 36 views of this sculpture at 10 degree intervals can be viewed on the author's home page, http://www.bgsu.edu/~long, thanks to his son, Andy Long, a mathematician, now a Senior Fulbright Scholar at the Institute of Mathematics and Physics in Porto Novo, Benin, West Africa. The eight knot basis for these sculptures was first visualized using piecewise special Bezier quartic curves [10] defined as an interpolating curve for the eight intersection points shown in fig. 7b (which stem from lines drawn from the center to the vertices of a regular tetrahedron). However the physical electrical wire knot shown here is an even greater help in actually carving the final surface. There are of course other surfaces which edge on this and other figure-eight knots [5].

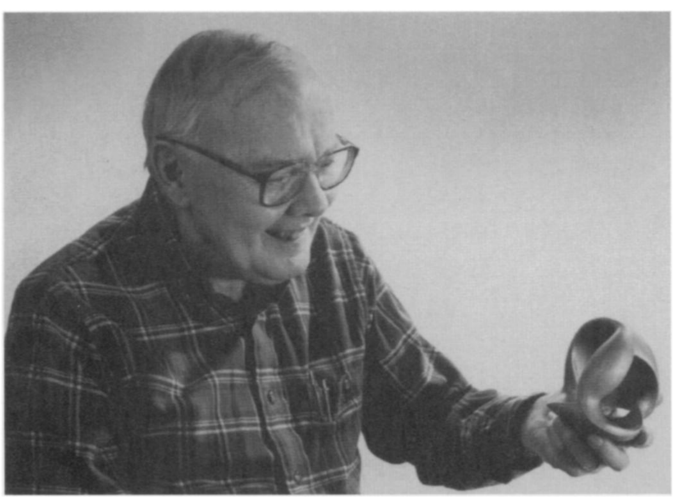

Figure 8. *The author with his favorite sculpture.*

A Special Minimal Surface

One of the author's most recent surface carvings (figure 9) provides for a clear understanding and physical image of "Costa's surface" ([8],[14]), a rather recently discovered surface. This surface developed from a set of equations given by Celso Costa and was shown by him to represent a complete minimal surface. It remained then for David Hoffman with the assistance of James Hoffman (no relation) and his computer software to show what this surface really looked like, and in the process show that it is not self-intersecting. Its discovery produced only the third of what is now known to be an infinite set of unusual surfaces referred to as complete, embedded minimal surfaces with finite total curvature. (The two previous ones, the plane and catenoid, being known for over two hundred years. The helicoid meets all of the conditions except finite total curvature.) It was the beautiful picture in the book by Ivars Peterson [14, p. 80] that introduced me to this surface and this reference is a good source for some of the terminology introduced in this paragraph. Once again, it was only after carving this crude wooden model that I really appreciated the surface itself. Incidentally, when milling a wax model for a surface related to Fubini's theorem for iterated integrals, it became obvious that there is an unexpected visual connection between the Fubini and Costa surfaces [8]. I only recently found out about an exhibit, "Beyond Numbers" at the Maryland Science Center in Baltimore, which includes information about and a large model of the Costa Surface. And I'm looking forward to seeing a video concern-

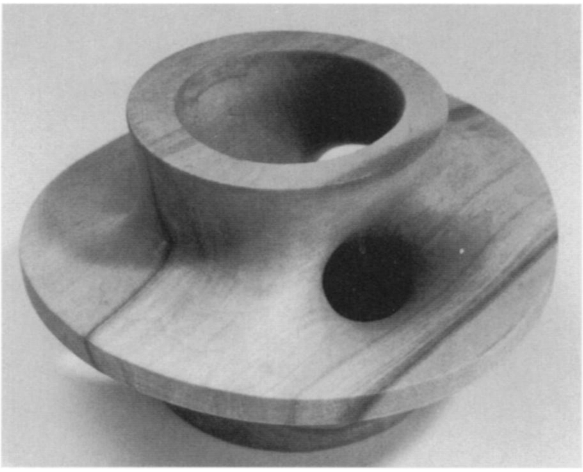

Figure 9

ing the surface produced by undergraduates at Bryn Mawr College under the direction of Professor Victor Donnay (Donnay@brynmawr.edu).

Create Your Own Models

The carvings and models shown here have significantly improved this author's understanding of the surfaces indicated and the mathematics behind them, as well as instilled interest in further study of related topics. One need not carve in wood: bring out the modeling clay and turn yourself loose. You may be amazed at what you'll learn in the process and what you can create for profit to others as well as yourself.

You may even be lucky enough to attend a Mathematics and Art Conference at SUNY Albany and participate in a stone carving session with mathematicians and master carvers Helaman Ferguson [1] and Nathaniel Friedman[4]. My experience with them led to the stone carving (in white alabaster) illustrated in fig. 10, of the previously mentioned orientable spanning surface of a trefoil knot.

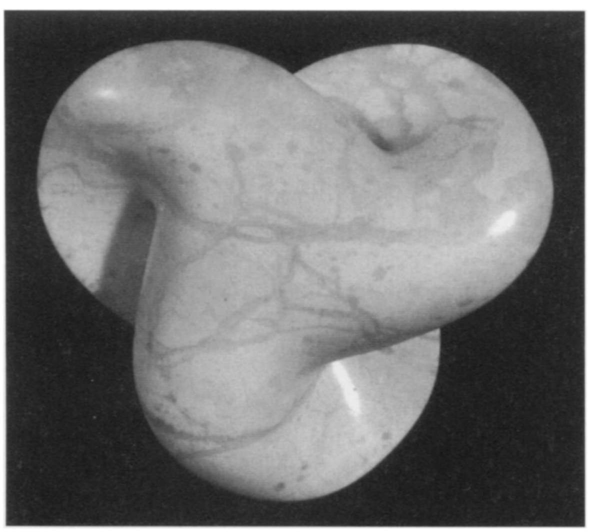

Figure 10

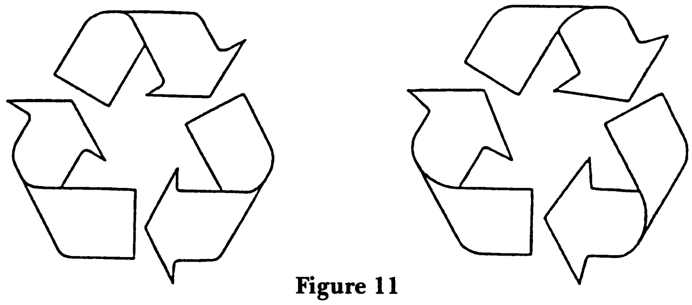

Figure 11

Perhaps I should warn you that once you start 3D modeling, you may start looking at two-dimensional drawings in a totally different way. For example, a few months ago I observed that there are at least two different recycling symbols in regular use—copies of which are shown in fig. 11. The first represents a spanning surface of a Mobius band and the second a spanning surface of a trefoil knot, both non-orientable [11]. □

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